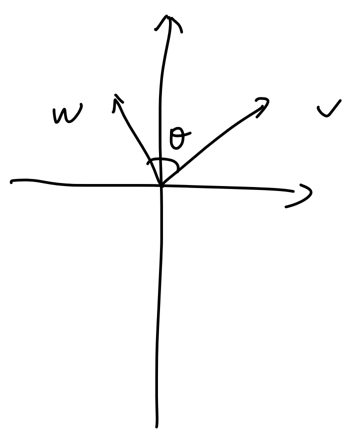
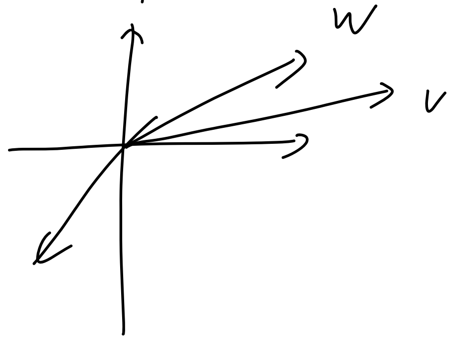
$\mathbb{R}^n$ 更多结构

$\mathbb{R}^2$




长度，角度

$\mathbb{R}^3$



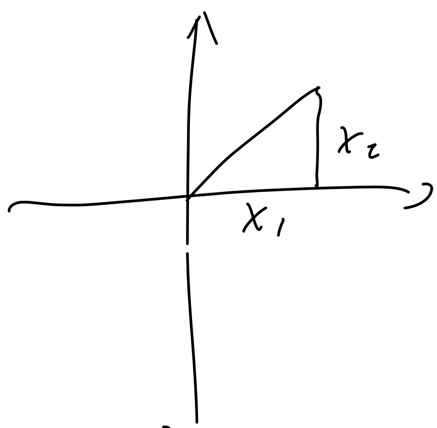

内积，点积

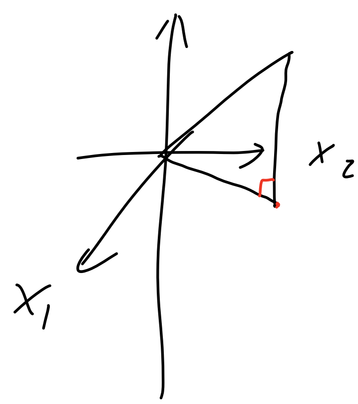
inner product

dot product.

定义： $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$. 长度

$$|x| = \sqrt{(x_1)^2 + (x_2)^2 + \cdots + (x_n)^2}$$




定义： $q(x) = |x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$

定义：内积 $g: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

$$(x, y) \mapsto g(x,y)$$

$$x = \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ \vdots \\ \vdots \\ y_n \end{pmatrix} \qquad = \sum_{i=1}^{n} x_i y_i$$

$$= x^T y$$

记号 $\langle x, y \rangle = g(x,y) = x^T y$.

$$q(x) = g(x, x)$$

性质：① $\langle \, , \, \rangle$ 双线性

$\forall c \in \mathbb{R}, \quad x, y, z \in \mathbb{R}^n$

$$\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$$

$$\langle x, c \cdot y \rangle = c \cdot \langle x, y \rangle$$

$$\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$$

$$\langle cx, y \rangle = c \cdot \langle x, y \rangle$$

② 对称性. $\langle x, y \rangle = \langle y, x \rangle$

③ $q(x) = \langle x, x \rangle \geq 0$ (正定性)

$$q(x) = 0 \iff x = 0$$

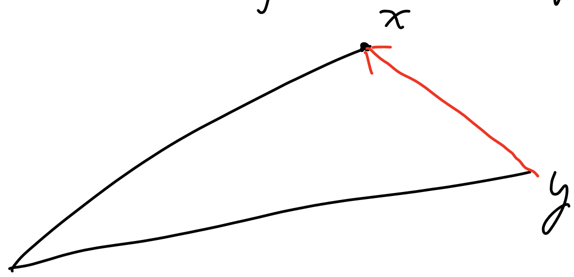
④ $\langle x, y \rangle = \frac{1}{2} \left( q(x+y) - q(x) - q(y) \right)$

验证.

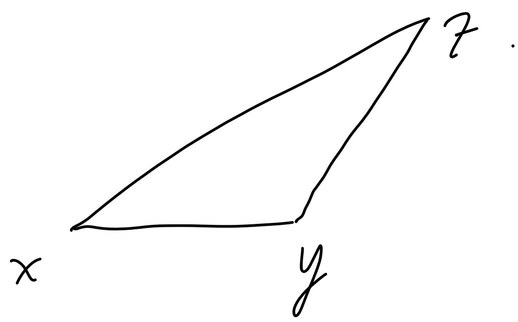
④ ⇒ "保持长度" ⇒ "保持内积".

定义: $\mathbb{R}^n$ 中两点之间的距离

$$\mathrm{dist}(x,y) \overset{\text{定义}}{=} \sqrt{q(x-y)} \geq 0$$




距离不等式（三角不等式）




$$\mathrm{dist}(x,y) + \mathrm{dist}(y,z) \geq \mathrm{dist}(x,z)$$

证明: $$\sqrt{q(\underbrace{x-y}_{X})} + \sqrt{q(\underbrace{y-z}_{Y})} \geq \sqrt{q(\underbrace{x-z}_{X+Y})}$$

$$\Leftrightarrow \sqrt{q(X)} + \sqrt{q(Y)} \geq \sqrt{q(X+Y)}$$

平方 $\iff$ $q(x)+q(y)+2\sqrt{q(x)q(y)}$

$\geq q(x+y)$

$\iff$ $|\langle x,y\rangle|^2 \leq \langle x,x\rangle\cdot\langle y,y\rangle$

性质：(柯西不等式)

Pf: $q(x+ty)\geq 0,\quad \forall t\in\mathbb{R}$

$t^2\langle y,y\rangle + 2t\langle x,y\rangle + \langle x,x\rangle \geq 0$

判别式 $\Delta \leq 0$

$\Rightarrow$ Cauchy 不等式.

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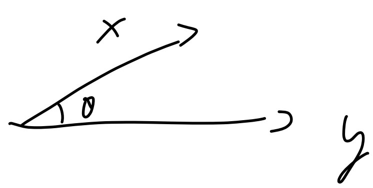
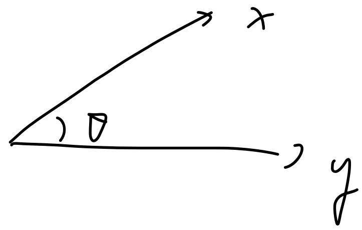
回顾：$\mathbb{R}^2,\ \mathbb{R}^3$

几何：$\langle x,y\rangle = |x|\cdot|y|\cdot\cos\theta$.



定义： 夹角θ，定义为

(为什么？)

$$\langle x,y\rangle = |x|\cdot|y|\cdot\cos\theta.$$

由Cauchy，$x\neq 0$，$y\neq 0$，

$$-1\le \frac{\langle x,y\rangle}{|x|\,|y|}\le 1$$

$x=0$，或 $y=0$，θ任意值。

定义：正交（垂直） $x\perp y$，x与y正交

定义为 $\langle x,y\rangle=0$

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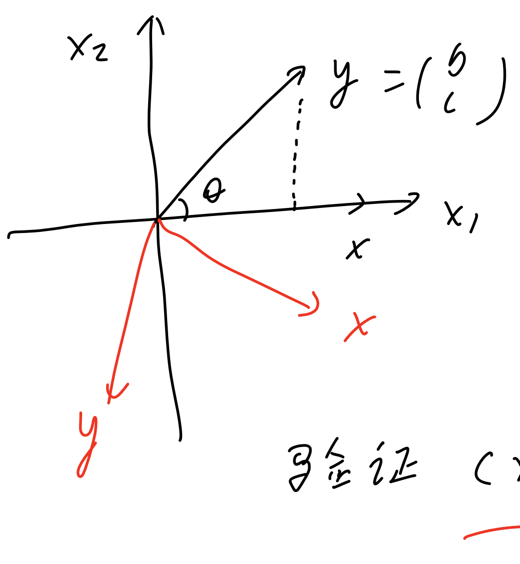
特殊情况：$\mathbb{R}^2$ $x=\begin{pmatrix}a\\0\end{pmatrix}$，$y=\begin{pmatrix}b\\c\end{pmatrix}$




$$\langle x,y\rangle = ab$$

$$b=|y|\cdot\cos\theta$$

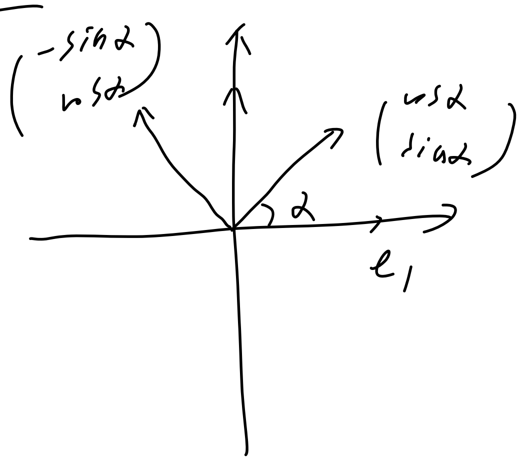
$$0\le\theta\le\frac{\pi}{2}$$

验证 $\langle x,y\rangle = |x|\cdot|y|\cdot\cos\theta.$

一般 $x,y\in\mathbb{R}^2$，通过"旋转"将x,y移动到特殊位置。

"旋转"不改变角度（几何上）

验证 旋转 不改变内积.（代数上）




逆时针转 $\alpha$

$$A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$\langle Ax, Ay \rangle = \langle x, y \rangle$$

$$\Leftrightarrow \quad (Ax)^T Ay = x^T y.$$

$$\Leftrightarrow \quad x^T (A^T A) y = x^T y.$$

$$A^T A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \cdot \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

定义（正交矩阵） $A \in M_n(\mathbb{R})$

$A^T A = I$ $\left( \begin{array}{l} \Leftrightarrow A^T = A^{-1} \\ \Leftrightarrow A \cdot A^T = I \end{array} \right)$

正交矩阵的集合 $O(n) = \{ A \in M_n(\mathbb{R}) \mid A^T A = I \}$

（可替换成 $q(Ax) = q(x)$，$\forall x \in \mathbb{R}^n$）

性质：$\forall x, y \in \mathbb{R}^n$，$\langle Ax, Ay \rangle = \langle x, y \rangle$

当且仅当 $A \in O(n)$

Pf："当" $\langle Ax, Ay \rangle = x^T (A^T A) y$

$= x^T y$

$= \langle x, y \rangle$

"仅当"，$\langle Ax, Ay \rangle = x^T (A^T A) y$

取 $x = e_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ ← $i$ 行 $=1$

$y = e_j$

$x^T \cdot (A^T A) \cdot y = A^T A$ $i$ 行 $j$ 列元素，

$$e_i^T \cdot e_j = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$$

$$\Rightarrow \quad A^T A = I_n$$

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线性变换角度：

定义：（正交变换）$T: \mathbb{R}^n \to \mathbb{R}^n$

$$\langle T(v), T(w) \rangle = \langle v, w \rangle \quad \forall v, w \in \mathbb{R}^n$$

性质：$T$ 是正交变换当且仅当

$T$ 在标准基 $B: e_1 \cdots e_n$ 下

$$[T]_B^B \in O(n)$$

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$T$ 在基上的作用决定了 $T$.

$$\underset{\substack{\| \\ v_1}}{T(e_1)} \cdots \underset{\substack{\| \\ v_n}}{T(e_n)}, \quad [T]_B^B = A$$

$$= [v_1 \cdots v_n]$$

找到 $A \in O(n)$ 关于 $v_1 \cdots v_n$ 的关系

$$A^TA = \begin{pmatrix} v_1^T \\ \vdots \\ v_n^T \end{pmatrix} \cdot (v_1 \cdots v_n)$$

$$= \left( v_i^T \cdot v_j \right)_{\substack{1\le i\le n \\ 1\le j\le n}}$$

$$= \left( \langle v_i, v_j \rangle \right)$$

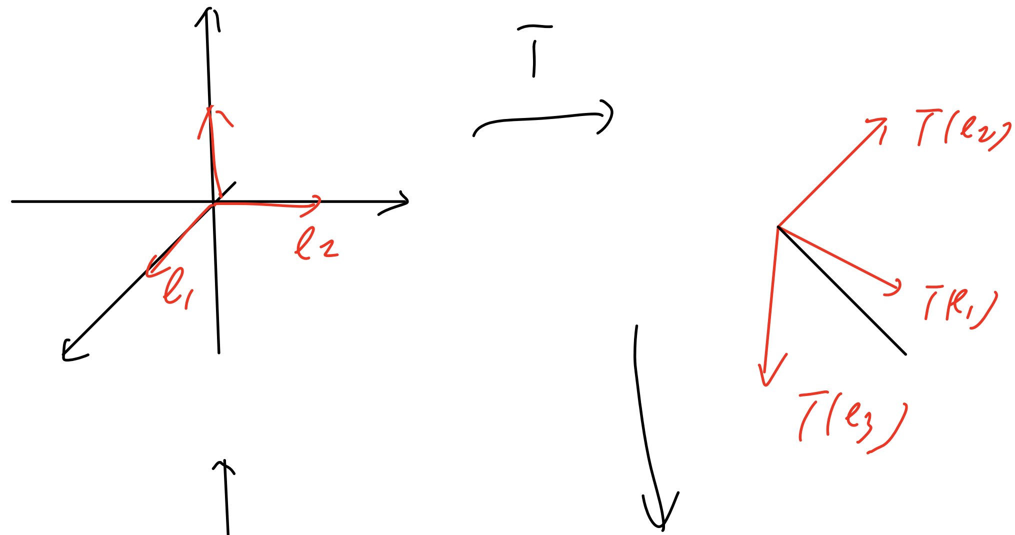
$A \in O(n) \iff$ (*) $\langle v_i, v_j \rangle = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$

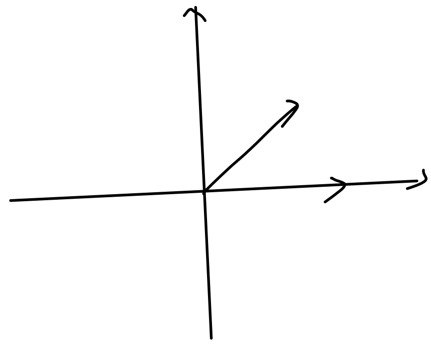
定义：基 $v_1 \cdots v_n$ 满足 (*)，称 $v_1 \cdots v_n$ 标准正交基。

（性质：$v_1 \cdots v_n$，$v_i \neq 0$，$\langle v_i, v_j \rangle = 0$，$i \neq j$。
$\Rightarrow v_1 \cdots v_n$ 线性无关）

性质：$A \in O(n)$，$\iff$ $v_1 \cdots v_n$ 是标准正交基。

性质'：$T$ 正交变换 $\iff$ $T$ 将标准正交基变成标准正交基



$B$ 是标准正交基

$|V|^2 = [V]_B$ 各分量

平方和.

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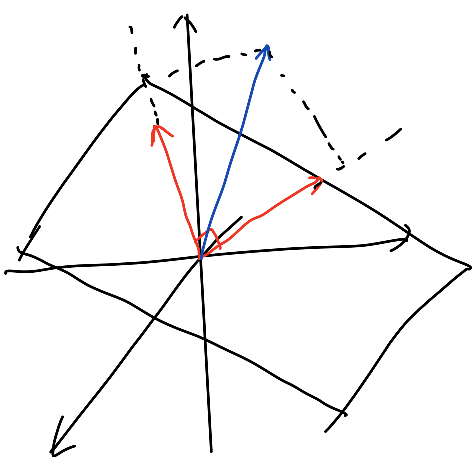
对 $W \subset \mathbb{R}^n$ 子空间, 可定义标准正交基

$B: w_1 \cdots w_n$ $W$基

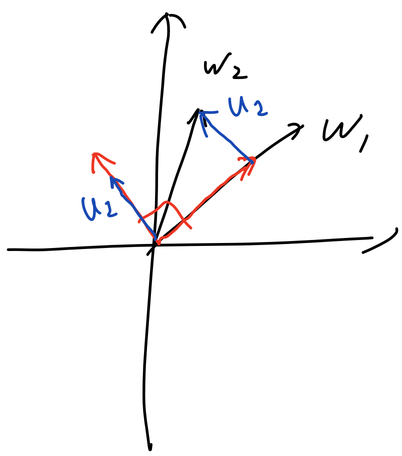
满足 $\langle w_i, w_j \rangle = \delta_{ij}$

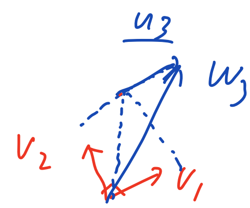


存在性：（Gram-Schmidt 正交化）

$w_1 \cdots w_m$ linearly independent $W = \text{span}(w_1, \ldots, w_m)$ 基

(G-S) $\Rightarrow$ 生成 $W$ 的标准正交基.




$$\begin{cases} u_1 = w_1, & v_1 = \dfrac{u_1}{|u_1|}, \quad \langle v_1, v_1 \rangle = 1 \\ u_2 = w_2 + c v_1 = w_2 - \langle w_2, v_1 \rangle v_1, & v_2 = \dfrac{u_2}{|u_2|}, \end{cases}$$

( $c$ 使得 $\langle u_2, v_1 \rangle = 0$.

$$\langle w_2 + c v_1, v_1 \rangle = 0$$

$$\langle w_2, v_1 \rangle + c \underbrace{\langle v_1, v_1 \rangle} = 0$$

$$c = - \langle w_2, v_1 \rangle$$
)

$$\langle v_2, v_1 \rangle = 0$$

$$\langle v_2, v_2 \rangle = 1$$

$u_3 = w_3 - \langle w_3, v_2\rangle v_2 - \langle w_3, v_1\rangle v_1$.

验证 $\langle u_3, v_2\rangle = 0$, $\langle u_3, v_1\rangle = 0$.

$v_3 = \dfrac{u_3}{|u_3|}$, $\langle v_3, v_2\rangle = 0$

$\langle v_3, v_1\rangle = 0$

$\langle v_3, v_3\rangle = 1$

$\vdots$

$u_m$. $v_m$.

$(v_1 \cdots v_m)$ 是 $\underbrace{\text{标准}}_{W}$正交基.

$$\underline{(\underline{w_1} \cdots w_m)} = \underline{(\underline{v_1} \cdots v_m)} \begin{pmatrix} \langle w_1, v_1\rangle & \langle w_2, v_1\rangle & \\ 0 & \langle w_2, v_2\rangle & \\ \vdots & \vdots & \\ 0 & 0 & \end{pmatrix}$$

上三角阵

推论: 任一标准正交向量组可扩充标准正交基.

$w_1 \cdots w_m \ \underline{v_{m+1} \cdots v_n}$ 扩充为基.

GS $w_1 \cdots w_m \ v'_{m+1} \cdots v'_n$

（非线性相关，$\underline{GS}$ 类似）

"矩阵"． $M \in M_{m\times n}(\mathbb{R})$． $\mathrm{rk}(M) = n$． $m \geq n$

$$M = \begin{pmatrix} w_1 & \cdots & w_n \\ & & \\ & & \end{pmatrix} \qquad \underline{w_i \in \mathbb{R}^m}$$

$$M = \underline{(v_1 \cdots v_n)} \cdot \begin{bmatrix} \diagdown & \\ 0 & \diagdown \end{bmatrix} \in \text{上三角阵}.$$

$$= Q_1 \cdot R_1 \qquad Q_1 \quad m \times n. \qquad R_1 \quad \underline{n\times n} \text{ 上三角}.$$

$$= Q \cdot R \qquad Q = (Q_1, Q_2) \in O(m)$$

$$R = \begin{pmatrix} R_1 \\ 0 \end{pmatrix}, \quad m \times n$$

称为 $QR$ 分解．

$\underline{m=n}$ 时，$A$ 可逆

$$A = \underset{\text{正交}}{\underline{Q}}\ \underset{\text{上三角阵．（对角线}>0\text{）}}{\underline{R}}$$

（唯一性）

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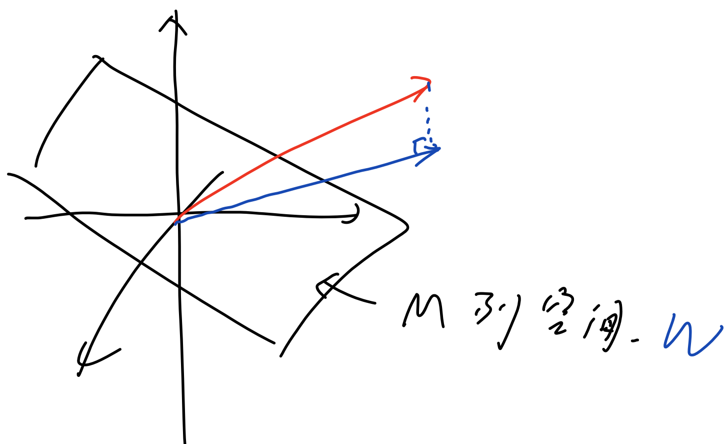
求解 $M\cdot x = b$，　$M$ $m\times n$.　$b\in\mathbb{R}^m$.

$rk\,M = n$　　$m\geq n$.

不一定有解.




找近似解：$|M\cdot x - b|$ 最小

$\Leftrightarrow\ Mx - b \perp W$

$$M = Q\cdot R = (Q_1\ \ Q_2)\cdot\begin{pmatrix}R_1\\0\end{pmatrix}$$

$$|Mx-b|^2 = |QRx - b|^2$$
$$= |Rx - Q^Tb|^2$$

$$= \left\| \begin{pmatrix} R_1 \cdot x \\ 0 \end{pmatrix} - \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \right\|^2$$

$$= \underbrace{\| R_1 \cdot x - C_1 \|^2}_{0} + \underline{\| C_2 \|^2}$$

$$R_1 = \begin{bmatrix} * & \ddots & * \\ 0 & & * \end{bmatrix}$$

$$\boxed{R_1 x = C_1}$$